Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 1, pp. 102–108, 2003

DETERMINATION OF TEMPERATURE DUE TO SLIP BETWEEN THE WHEEL AND THE RAIL TAKING INTO ACCOUNT CONVECTIVE COOLING OF FREE SURFACES

A. A. Evtushenko[1] and S. Ya. Matysyak[2]

UDC 536.12:621.891

A computational algorithm for determining temperature due to slip between the train wheel and the rail is proposed. The algorithm uses Ling's solution of the mixed two-dimensional quasisteady-state thermal-conductivity problem for a half-space heated locally by a fast moving distributed heat flow. This solution is calculated using the method of piecewise linear approximation by finite functions. An analytical solution of the problem is obtained for the particular case of uniform distribution of the frictional heat flow rate. The effect of different forms of heat flow rate distribution and the Biot number on the rail temperature field is studied.

Key words*: friction, heat flow, contact pressure, Volterra integral equation.*

Introduction. Friction in the contact area between the wheel and the rail leads to conversion of mechanical to thermal energy [1]. Frictional heating results in a local temperature rise in the contact area of rubbing bodies and can bring about changes in the microstructure of the body materials with subsequent failure. To determine the heating temperature due to slip between the wheel and the rail, Harrison [2] proposed using the results Archard [3], who obtained a solution of the steady-state thermal-conductivity problem for a half-space whose finite surface region is heated by a uniformly distributed frictional heat flow. Jaeger [4] assumed that the heated region in the indicated problem is shaped like a square or a circle.

For the corresponding quasisteady-state thermal-conductivity problem for a half-space heated by a fast moving linear heat flow, a solution in the form of a convolution integral of the heat flow rate and a kernel with a root singularity was obtained by Ling and Mow [5] using an integral Fourier transform. Tanvir [6] and Knothe and Liebelt [7] studied the case of elliptic distribution of frictional heat flow rate proportional to Hertz contact pressure using an integral Laplace transform. Evtushenko and Semerak [8] solved the problem by approximate integration using piecewise linear functions.

In all studies cited above, it was assumed that the surface of the half-space outside the heating region was heat insulated. The goal of the present study is to obtain a solution of the Ling thermal problem [5] for an arbitrary heat flow rate taking into account convective cooling of the free surface of the half-space and to simulate the thermal regime of the wheel-rail tribosystem.

1. Formulation of the Problem. We consider a circular cylinder of radius R (wheel) which moves uniformly with translational velocity V over the surface of a half-space (rail) (Fig. 1). The cylinder is pressed into the half-space surface by linear force P. In the contact area between the wheel and the rail, slippage with velocity $V_{\rm s}$ leads to heat generation in the form of heat flows directed into the wheel and the rail. Assuming that the heat flow distribution coefficient is known [9], we consider the temperature distribution in the rail due to friction. To this end, we use a Cartesian coordinate system xy rigidly attached to the leading edge of the wheel. The following assumptions are adopted:

---

[1]Podstrigach Institute of Applied Problems of Mathematics and Mechanics, National Academy of Sciences of Ukraine, L'vov 70053, Ukraine; [2]Warsaw University, Warsaw 02089, Poland. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 1, pp. 123–130, January–February, 2003. Original article submitted March 6, 2002.

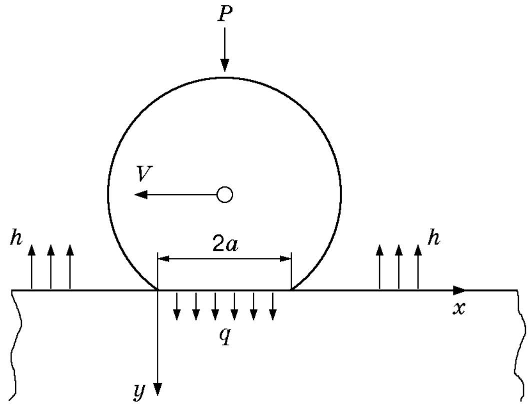

Fig. 1

— the length $2a$ of the contact area is small compared to the wheel radius R;

— for creep varying in the range $0.1\% < s < 2\%$ (macroslip), the slip velocity V_s is equal to [6]

$$V_s = sV; \tag{1}$$

— the rail surface is heated by a fast moving distributed heat flow [10], whose gradient in the slip direction is negligible compared to the gradient in the transverse direction;

— the frictional heat flow rate q is proportional to the friction power:

$$q(x) = \gamma f V_s p(x), \qquad 0 \leqslant x \leqslant 2a$$

where p is the contact pressure, f is the friction coefficient, and γ is the heat flow distribution coefficient;

— convective cooling of the rail surface occurs outside the heating region;

— the temperature state of the rail in the Euler coordinates xy is steady-state;

— the thermal properties of the rail material are constant.

Under the above assumptions, the thermal problem for the rail reduces to the following quasisteady-state thermal-conductivity problem for the half-space:

$$\frac{\partial^2 T}{\partial \eta^2} = \frac{\partial T}{\partial \xi}, \qquad -\infty < \xi < \infty, \quad 0 \leqslant \eta < \infty,$$

$$\frac{\partial T}{\partial \eta}\Big|_{\eta=0} = \begin{cases} -\Lambda p^*(\xi), & 0 \leqslant \xi \leqslant 1, \\ \mathrm{Bi}\, T, & -\infty < \xi < 0 \vee 1 < \xi < \infty, \end{cases} \qquad T \to 0 \text{ for } \sqrt{\xi^2 + \eta^2} \to \infty. \tag{2}$$

Here

$$\Lambda = \frac{\gamma f V_s p_0 d}{K}, \quad d = \sqrt{\frac{2ak}{V_s}}, \quad p_0 = \frac{2P}{\pi a}, \quad \mathrm{Bi} = \frac{hd}{K}, \quad \xi = \frac{x}{2a}, \quad \eta = \frac{y}{d}, \tag{3}$$

T is the temperature, K and k are thermal conductivity and thermal diffusivity, respectively, h is the heat-transfer coefficient, d is the effective heating depth in the corresponding unsteady thermal-conductivity problem [7], and p_0 is the maximum pressure in the two-dimensional analog of Hertz's problem [11].

2. Method of Solution. The solution of the quasisteady-state thermal conductivity problem (2) for arbitrary distribution of the dimensionless contact pressure $p^* = p/p_0$ is written in the form [5]

$$T(\xi, \eta) = T_{\max} T^*(\xi, \eta), \qquad -\infty < \xi < \infty, \quad 0 \leqslant \eta < \infty,$$

where

$$T^*(\xi, \eta) = \frac{2}{\pi} \int_0^b p^*(\tau) G(\xi - \tau, \eta)\, d\tau - H(\xi - 1) \frac{\mathrm{Bi}}{\sqrt{\pi}} \int_b^\xi T^*(\lambda) G(\xi - \lambda, \eta)\, d\lambda; \tag{4}$$

$$G(\xi, \eta) = \frac{e^{-\eta^2/(4\xi)}}{\sqrt{\xi}}, \qquad b = \begin{cases} 0, & -\infty < \xi \leqslant 0, \\ \xi, & 0 \leqslant \xi \leqslant 1, \\ 1, & 1 \leqslant \xi < \infty, \end{cases} \tag{5}$$

$H(\cdot)$ is the unit Heaviside function, $T^*(\lambda) \equiv T^*(\lambda, 0)$ and $T_{\max} = \Lambda\sqrt{\pi}/2$ is the maximum temperature at the point $\xi = 1$ at the exit from the contact area in the case of uniform contact pressure distribution [8]

$$p(x) = \pi p_0/4, \qquad 0 \leqslant x \leqslant 2a \tag{6}$$

and heat insulation of the rail surfaces outside the friction region, i.e., at $h = 0$ ($\mathrm{Bi} = 0$).

To calculate the integrals on the right side of solution (4), we employ the method of piecewise linear approximation by finite functions: [12]. Let us introduce a uniform mesh on the integration interval $[0, b]$:

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = b, \quad \tau_i = i\delta\tau, \quad \delta\tau = b/n, \quad i = 0, 1, \ldots, n.$$

Each node τ_i is put in correspondence to a "cover function":

$$\varphi_0(\tau) = \begin{cases} (\tau_1 - \tau)/(\delta\tau), & \tau \in [\tau_0, \tau_1], \\ 0, & \tau \notin [\tau_0, \tau_1], \end{cases} \qquad \varphi_n(\tau) = \begin{cases} (\tau - \tau_{n-1})/(\delta\tau), & \tau \in [\tau_{n-1}, \tau_n], \\ 0, & \tau \notin [\tau_{n-1}, \tau_n], \end{cases}$$

$$\varphi_i(\tau) = \begin{cases} (\tau - \tau_{i-1})/(\delta\tau), & \tau \in [\tau_{i-1}, \tau_i], \\ (\tau_{i+1} - \tau)/(\delta\tau), & \tau \in [\tau_i, \tau_{i+1}], \\ 0, & \tau \notin [\tau_{i-1}, \tau_{i+1}], \end{cases} \qquad i = 1, 2, \ldots, n-1. \tag{7}$$

The dimensionless contact pressure $p^*(\tau)$ is approximated by means of piecewise linear functions (7):

$$p^*(\tau) \simeq \sum_{i=0}^{\infty} p_i^* \varphi_i(\tau), \qquad p^* \equiv p^*(\tau_i). \tag{8}$$

The uniform error of this approximation has order $O(\delta\tau^2)$ [13].

Substituting expansion (8) into Eq. (4) for $\mathrm{Bi} = 0$ and integrating, we obtain

$$T^*(\xi, \eta) = \frac{2}{\pi\delta\tau} \sum_{i=0}^{n} p_i^* G_i(\xi, \eta) H(\xi), \qquad -\infty < \xi < \infty, \quad 0 \leqslant \eta < \infty, \tag{9}$$

where

$$\begin{gathered} G_0(\xi, \eta) = \tau_1 G_1^{(0)}(\xi, \eta) - G_1^{(1)}(\xi, \eta), \qquad G_n(\xi, \eta) = G_n^{(1)}(\xi, \eta) - \tau_{n-1} G_n^{(0)}(\xi, \eta), \\ G_i(\xi, \eta) = G_i^{(1)}(\xi, \eta) - \tau_{i-1} G_i^{(0)}(\xi, \eta) + \tau_{i+1} G_{i+1}^{(0)} - G_{i+1}^{(1)}(\xi, \eta), \qquad i = 1, 2, \ldots, n-1, \\ G_i^{(0)}(\xi, \eta) = F^{(0)}(\xi - \tau_i, \eta) - F^{(0)}(\xi - \tau_{i-1}, \eta), \qquad \xi \geqslant \tau_i, \quad i = 1, \ldots, n, \\ G_i^{(1)}(\xi, \eta) = F^{(1)}(\xi - \tau_i, \eta) - F^{(1)}(\xi - \tau_{i-1}, \eta) + (\eta^2/6 + \xi) G_i^{(0)}(\xi, \eta), \\ F^{(0)}(\xi, \eta) = -2\sqrt{\xi}\, \mathrm{e}^{-\eta^2/(4\xi)} - \eta\sqrt{\pi}\, \mathrm{erf}\,(0.5\eta/\sqrt{\xi}), \quad F^{(1)}(\xi, \eta) = 2\xi\sqrt{\xi}\, \mathrm{e}^{-\eta^2/(4\xi)}/3 \end{gathered} \tag{10}$$

[erf $(\cdot)$ is an error function].

The dimensionless surface temperature $T^*(\lambda)$ ($\lambda \geqslant 1$) in the second term of the right side of the integral equation (4)is written as

$$T^*(\lambda) \simeq \sum_{j=0}^{m} T_j^* \varphi_j(\lambda), \qquad T_j^* \equiv T^*(\lambda_j), \tag{11}$$

where $\lambda_j = 1 + j\delta\lambda$ ($j = 0, 1, \ldots, m,$), $\delta\lambda = (\xi - 1)/m$, and $\varphi_i(\lambda)$ is the "cover function" (7).

Substitution of expansion (11) taking into account solution (9) into Eq. (4) yields the following system of linear algebraic equations with a triangular matrix:

$$T_k^* + \frac{\mathrm{Bi}}{\sqrt{\pi}\,\delta\lambda} \sum_{j=0}^{k} T_j^* G_{jk} = \frac{2}{\pi\delta\tau} \sum_{i=0}^{n} p_i^* G_{ik}, \qquad G_{jk} \equiv G_j(\lambda_k, 0), \quad k = 0, 1, \ldots, m. \tag{12}$$

Solution of system (12) gives the dimensionless surface temperature T_j^* at the nodes λ_j ($j = 0, 1, \ldots, m$). Substituting these values into the integral equation (4), we obtain the dimensionless temperature field in the rail behind the contact area:

$$T^*(\xi, \eta) = \frac{2}{\pi\delta\tau} \sum_{i=0}^{n} p_i^* G_i(\xi, \eta) - \frac{\mathrm{Bi}}{\sqrt{\pi}\,\delta\lambda} \sum_{j=0}^{m} T_j^* G_j(\xi, \eta), \qquad \xi \geqslant 1, \quad \eta \geqslant 0. \tag{13}$$

3. Uniform Contact Pressure Distribution. We consider the case of a uniform pressure distribution, typical of heavily loaded friction units [14]. Substituting the dimensionless contact pressure $p^*(\tau)$ (6) into Eq. (4) and integrating the result, we obtain

$$T(\xi,\eta) = \begin{cases} 0, & -\infty < \xi \leqslant 0, \\ \theta(\xi,\eta), & 0 \leqslant \xi \leqslant 1, \\ \theta(\xi,\eta) - \theta(\xi-1,\eta) - \dfrac{\mathrm{Bi}}{\sqrt{\pi}} \displaystyle\int_1^{\xi} T^*(\lambda) G(\xi-\lambda,\eta)\, d\lambda, & 1 \leqslant \xi < \infty, \end{cases} \tag{14}$$

where

$$\theta(\xi,\eta) = \sqrt{\xi}\, \mathrm{e}^{-\eta^2/(4\xi)} - \sqrt{0.5\pi}\, \eta\, \mathrm{erfc}\,(0.5\eta/\sqrt{\xi}), \quad 0 \leqslant \eta < \infty, \qquad \mathrm{erfc}\,(\cdot) = 1 - \mathrm{erf}\,(\cdot). \tag{15}$$

From relations (14) it follows that in order to determine the surface temperature $T^*(\lambda)$ behind the friction region $\xi \geqslant 1$, $\eta = 0$, it is necessary to solve the Volterra integral equation of the second kind with a weakly singular kernel:

$$T^*(\xi) + \frac{\mathrm{Bi}}{\sqrt{\pi}} \int_1^{\xi} \frac{T^*(\lambda)}{\sqrt{\xi-\lambda}}\, d\lambda = F(\xi), \qquad 1 \leqslant \xi < \infty, \tag{16}$$

where $F(\xi) = \sqrt{\xi} - \sqrt{\xi - 1}$.

The solution of the integral equation (16) is written in the form [15]

$$T^*(\xi) = F(\xi) + \int_1^{\xi} R(\xi-\tau) F(\tau)\, d\tau, \qquad 1 \leqslant \xi < \infty, \tag{17}$$

where R is the resolvent:

$$R(\xi) = \sum_{n=1}^{\infty} \frac{(-\mathrm{Bi}\sqrt{\xi})^n}{\xi \Gamma(n/2)} \tag{18}$$

[$\Gamma(\cdot)$ is the gamma function]. For $n = 2k$, from relation (18) we have

$$R_{2k}(\xi) = \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}^2\, \xi)^k}{\xi \Gamma(k)} = \mathrm{Bi}^2 \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}^2\, \xi)^{k-1}}{(k-1)!} = \mathrm{Bi}^2\, \mathrm{e}^{\mathrm{Bi}^2\, \xi}, \tag{19}$$

and for $n = 2k+1$, taking into account formula 5.2.7.18 of [16], we obtain

$$R_{2k+1}(\xi) = \sum_{k=0}^{\infty} \frac{(-\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\xi \Gamma(k+0.5)} = \frac{-\mathrm{Bi}\sqrt{\xi}}{\xi \Gamma(0.5)} - \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\xi \Gamma(k+0.5)}$$

$$= -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} - \mathrm{Bi}^2 \sum_{k=0}^{\infty} \frac{(\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\Gamma(k+1.5)} = -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} - \mathrm{Bi}^2\, \mathrm{e}^{\mathrm{Bi}^2\, \xi}\, \mathrm{erf}\,(\mathrm{Bi}\sqrt{\xi}). \tag{20}$$

Summing relations (19) and (20), we obtain

$$R(\xi) = -\mathrm{Bi}/\sqrt{\pi\xi} + \mathrm{Bi}^2\, \mathrm{e}^{\mathrm{Bi}^2\, \xi}\, \mathrm{erfc}\,(\mathrm{Bi}\sqrt{\xi}). \tag{21}$$

Since the integral in relation (17) with the resolvent (21) cannot be calculated analytically, we use the asymptotic expansions of the resolvent [17]

$$R(\xi) \simeq \tilde{R}(\xi) = -\mathrm{Bi}/\sqrt{\pi\xi} + \mathrm{Bi}^2(1 - 2\,\mathrm{Bi}\,\xi/\pi + \mathrm{Bi}^2\, \xi), \qquad \mathrm{Bi}\sqrt{\xi} \ll 1; \tag{22}$$

$$R(\xi) \simeq \tilde{\tilde{R}}(\xi) = -1/(2\,\mathrm{Bi}\,\xi\sqrt{\pi\xi}), \qquad \mathrm{Bi}\sqrt{\xi} \gg 1. \tag{23}$$

For wheel–rail tribosystems, the width of the contact area is $2a \approx 0.01$ m and the heat-transfer coefficient is $h = 0$–200 W/(m$^2\cdot$ K) [17]. Using thermal steady-state constants for steel $K = 41$ W/(m$\cdot$ K) and $k = 9.1 \cdot 10^{-6}$ m^2/sec, we find that the Biot criterion varies within $0 \leqslant \mathrm{Bi} \leqslant 0.15 \cdot 10^{-2}/\sqrt{V_{\mathrm{s}}}$. According to formula (1), for translational motion of the wheel at velocity $V \approx 75$ m/sec and creep $s = 1$ %, the slip velocity is $V_{\mathrm{s}} \approx 0.75$ m/sec. Therefore, the upper bound of the Biot criterion does not exceed 0.01. Numerical study of the behavior

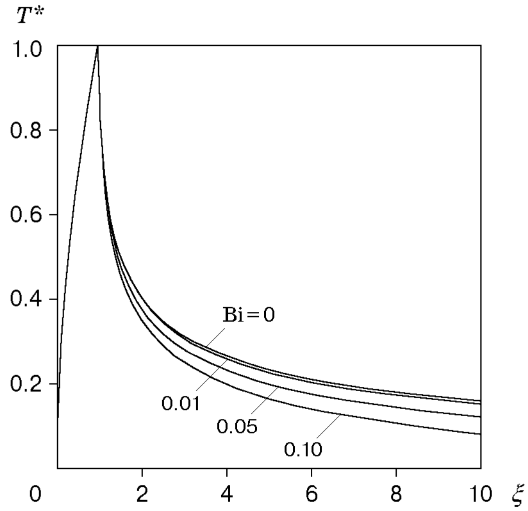

Fig. 2

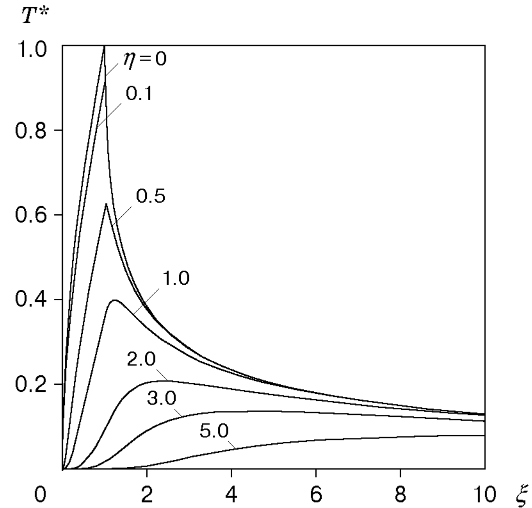

Fig. 3

of the resolvent R showed that the absolute error of its approximation using the asymptotic relation (22) does not exceed 1 % for values of the argument $\xi - \tau \leqslant \mathrm{Bi}^{-2}$.

Thus, the integral on the right side of relation (17) can be evaluated by using the asymptotic expression (22) for $\tilde{R}$ at $1 \leqslant \xi \leqslant 10^4$. As a result, we find the rail surface temperature behind the contact area

$$T^*(\xi) = \sqrt{\xi} - \sqrt{\xi - 1} - \mathrm{Bi}\,\theta_1(\xi), \qquad 1 < \xi < \infty, \tag{24}$$

where

$$\theta_1(\xi) = (1/\sqrt{\pi})[\sqrt{\xi - 1} - 0.5\pi(\xi - 1) + \xi \arcsin\sqrt{1 - 1/\xi}] - (2/3)\,\mathrm{Bi}[\xi\sqrt{\xi} - 1 - (\xi - 1)\sqrt{\xi - 1}]$$
$$+ [\mathrm{Bi}^2/(2\sqrt{\pi})][(\xi - 2)\sqrt{\xi - 1} - 0.5\pi(\xi - 1)^2 + \xi^2 \arcsin\sqrt{1 - 1/\xi}]$$
$$- (2\,\mathrm{Bi}^3/15)[3 - 5\xi + 2\xi^2\sqrt{\xi} - 2(\xi - 1)^2\sqrt{\xi - 1}\,]. \tag{25}$$

Knowing the surface temperature [see (24) and (25)], we find the temperature field at an arbitrary point of the rail behind the contact area from the relation

$$T(\xi, \eta) = \theta(\xi, \eta) - \theta(\xi - 1, \eta) - \frac{\mathrm{Bi}}{\sqrt{\pi}\delta\lambda} \sum_{j=0}^{m} T_j^* G_j(\xi, \eta), \qquad \xi > 1, \quad \eta > 0, \tag{26}$$

where the functions θ and G_j have the form of (15) and (10), respectively.

4. Numerical Analysis and Conclusions. Calculations were performed for the dimensionless temperature T^*. The dimensionless initial parameters are the coordinates ξ and η, the Biot criterion Bi, and the numbers n and m of points of division of the intervals $[0; b]$ and $[1; \xi]$, respectively. The parameters n and m were chosen so as to reach the required calculation accuracy.

We first study the case of uniform contact pressure distribution (6). Calculations using formulas (14), (15), and (24)–(26) showed that the temperature behind the contact area decreases with increase in Bi (Fig. 2). The temperature in the heating region does not depend on the Biot criterion, and its maximum value is reached at the point $\xi = 1$ on the rail surface that separates the heating and cooling regions.

For $\mathrm{Bi} = 0.05$, the temperature decreases rapidly with distance from the rail surface (Fig. 3). This process proceeds fastest above the heating region $0 \leqslant \xi \leqslant 1$. In thermal calculations of friction units, an important characteristic is the effective heating depth [18] , i.e., the distance from the slip surface, on which the temperature accounts for 5% of the maximum surface temperature. For the sections $\xi = 1$, 2, and 5, the effective depths is equal to $3d$, $5d$, and $7d$, respectively, i.e., it increases with distance from the heating region.

Most often, engineering calculations use an elliptic distribution (Hertz's distribution) of contact pressure [11]

$$p(x) = p_0 p^*(x), \qquad p^*(x) = \sqrt{1 - ((x - a)/a)^2}, \qquad 0 \leqslant x \leqslant 2a. \tag{27}$$

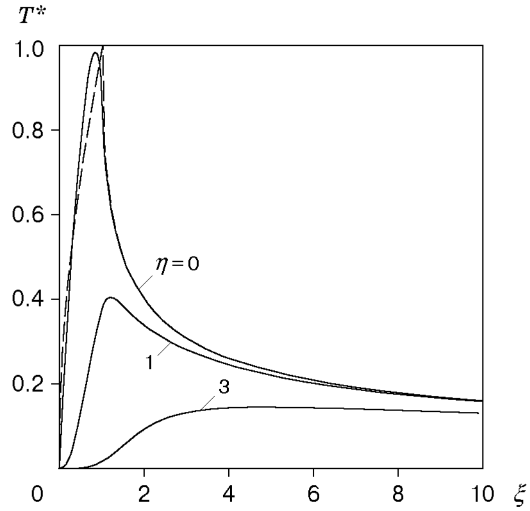

Fig. 4

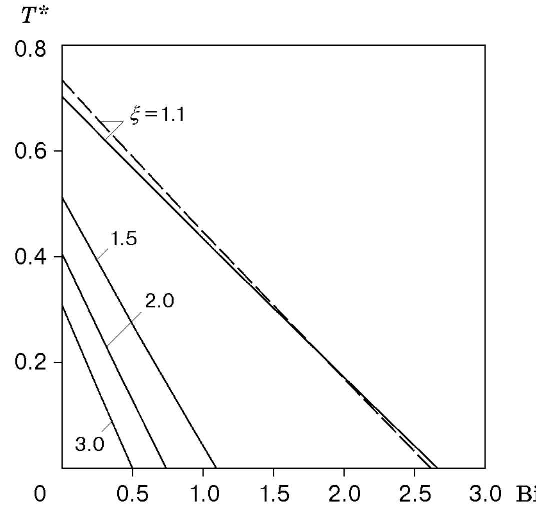

Fig. 5

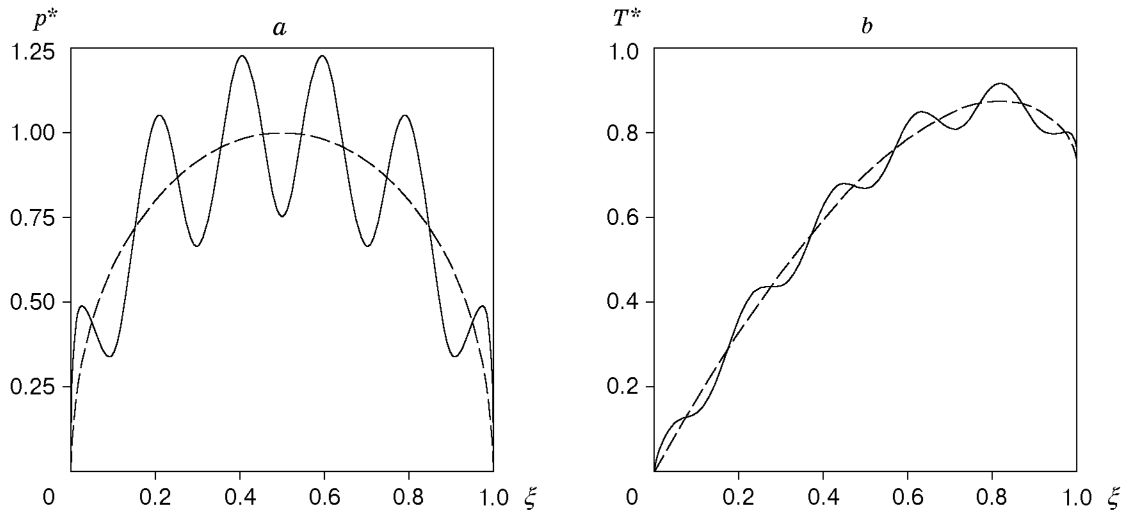

Fig. 6

Figure 4 gives a curve of $T^*(\xi)$ for Bi = 0.01 (solid curves refer to calculations using an elliptic pressure distribution and dashed curves refer to a uniform contact pressure distribution). Calculations by formulas (12) and (13) showed that a significant difference in the temperature distributions for relations (6) and (27) is observed only in the contact area. Behind this area, the corresponding temperatures are practically equal. Therefore, the rail surface temperature behind the heating region for the elliptic contact pressure distribution (27) can be calculated using the analytical solution (24), (25), obtained for the case of a uniform pressure distribution (6).

The dependence of the rail surface temperature T^* on the Biot criterion is linear (Fig. 5). An insignificant difference in curves of $T^*(\mathrm{Bi})$ for the constant (6) (dashed curve) and elliptic (27) (solid curves) contact pressure distributions is observed only in the immediate vicinity ($1.0 \leqslant \xi \leqslant 1.1$) of the contact area.

We also studied the effect of rail surface roughness on surface temperature. The distributions of contact pressure, and hence, the frictional heat flow rate were taken as a superposition of the elliptic and oscillating pressures:

$$p(x) = p_0[\sqrt{1 - ((x-a)/a)^2} - (1/4)\cos(5\pi(x-a)/a)], \qquad 0 \leqslant x \leqslant 2a. \tag{28}$$

Figure 6a and b shows distributions of the dimensionless elliptic (27) (dashed curve) and oscillating (28) (solid curves) pressures and the corresponding dimensionless surface temperatures. It should be noted that if the maximum contact pressures for the indicated distributions differ by approximately 25%, the corresponding maximum temperatures differ by only 6%. Therefore, the rail surface roughness exerts an effect mainly on the contact pressure and, to a lesser extent, on the surface temperature.

REFERENCES

1. F. P. Bowden and D. Tabor, in: *The Friction and Lubrication of Solids*, Parts 1 and 2, Clarendon Press, Oxford (1950–1964).
2. D. Harrison, "Corrugation of railway rails," Ph. D. Thesis, Cambridge (1979).
3. J. F. Archard, "The temperature rise of rubbing surfaces," *Wear*, **2**, No. 6, 438–455 (1958–1959).
4. Jaeger J. C. "Moving sources of heat and the temperature of sliding contacts," *Proc. Roy. Soc. New South Wales*, **76**, Part 3, 203–224 (1942).
5. F. F. Ling and V. C. Mow, "Surface displacements of a convective half-space under an arbitrarily distributed high-speed heat source," *Trans. ASME, J. Basis Engng.*, **87**, No. 3, 729–734 (1965).
6. M. A. Tanvir, "Temperature rise due to slip between wheel and rail. An analytical solution for Hertzian contact," *Wear*, **61**, Nos. 1–2, 295–308 (1980).
7. K. Knothe and S. Liebelt, "Determination of temperatures for sliding contact with applications for wheel-rail systems," *Wear*, **189**, Nos. 1–2, 91–99 (1995).
8. A. A. Evtushenko and V. M. Semerak, " Determination of temperature due to slip between the wheel and the rail," *Trenie Iznos*, **20**, No. 6, 588–594 (1999).
9. O. V. Pereverzeva and V. A. Balakin, "Distribution of heat between rubbing bodies," *Trenie Iznos*, **13**, No. 3, 507–516 (1992).
10. V. C. Mow and H. S. Cheng, "Thermal stresses in an elastic half-space associated with an arbitrarily distributed moving heat sources," *Z. Angew. Math. Phys.*, **18**, No. 4, 500–507 (1967).
11. K. L. Johnson, *Contact Mechanics*, Cambridge Univ. Press, London (1985).
12. A. A. Yevtushenko and V. J. Pauk, "The plane transient contact problem for rough sliding bodies with wear and heat generation," *Wear*, **173**, No. 1, 39–50 (1994).
13. G. I. Marchuk and V. I. Agoshkov, *Introduction to Grid Projection Methods* [in Russian], Nauka, Moscow (1981).
14. I. V. Kragel'skii, M. M. Dobychin, and V. S. Kombalov, *Fundamentals of Friction and Wear Calculations* [in Russian], Mashinostroenie, Moscow (1977).
15. V. I. Smirnov, *Course of Higher Mathematics* [in Russian], Nauka, Moscow (1967).
16. I. I. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).
17. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, John Wiley and Sons, New York (1972).
18. A. V. Chichinadze, E. D. Brown, A. G. Ginsburg, and E. V. Ignat'eva, *Calculation, Testing, and Selection of Fictional Pairs* [in Russian], Nauka, Moscow (1979).